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A COMPARATIVE STUDY OF NUMERICAL METHODS FOR SOLVING PARTIAL DIFFERENTIAL EQUATIONS

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Abstract

This study compares and contrasts the numerical techniques used to solve partial differential equations (PDEs) with the goal of determining how effective these techniques are in different computational settings. In many branches of science and engineering, partial differential equations are used to describe complex systems that change across time and space. The study thoroughly analyses and contrasts well-known numerical methods, including spectral approaches, finite difference, and finite element, explaining their advantages, disadvantages, and suitability for various kinds of partial differential equations. The work provides insight into the stability, accuracy, and computational efficiency of these approaches under different scenarios through extensive simulations and benchmarks. As a study of communication that links the natural sciences and humans, mathematics presents all laws and issues as formulae and looks for solutions. The differential equation that will be covered in this dissertation is a component of mathematics that is commonly employed in all sciences. There is a different way to solve each of these equations; we have previously covered the calculus analytic approaches. Now, we will introduce the numerical solutions. It is important to remember that analytical methods cannot solve every problem. For those equations that cannot be solved analytically, scientists have found numerical solutions through these approaches. To familiarize the reader with these definitions and concerns prior to the process commencing, we first explain differential equations in this article along with some basic subjects.

Keywords:*Numerical Methods,Solving Partial,Equations,Mathematical Techniques*

1. INTRODUCTION

A key component of mathematical modelling in a wide range of scientific and engineering fields is the solution of partial differential equations, or PDEs. PDEs are essential to disciplines including physics, engineering, biology, and finance because they explain how physical events change over time and space. Since many PDEs have inherent complexity, finding analytical solutions can be difficult and requires the application of numerical techniques to get approximations. The accuracy, stability, and computational efficiency of the solution are all directly impacted by the numerical approach chosen, thus choosing the right one is essential.

This drives our thorough analysis of how well different numerical approaches to PDE solving perform in comparison.

Over the years, there have been major breakthroughs in numerical approaches for PDEs, with numerous strategies created to address various PDE problem types. Through a thorough comparison investigation, this study seeks to advance our understanding of various numerical techniques. Three popular approaches—finite difference, finite element, and spectral methods—will be highlighted. These methods differ in their fundamental concepts and discretization procedures, leading to unique trade-offs in terms of accuracy and processing cost. This comparative analysis is relevant because it can shed light on the advantages and disadvantages of each numerical method in different situations. We hope to assist practitioners and researchers in choosing the right numerical tools for certain applications by methodically assessing their performance on a variety of PDE problems. In order to solve the difficulties presented by more complicated PDEs and take use of advances in computational resources, the study also examines current innovations in adaptive mesh refinement and parallel computing. By means of this study, we hope to further the continuous improvement of numerical methods for solving PDEs and expand their usefulness in various fields of science and engineering.

1.1 Background

Partial differential equations (PDEs) are essential tools for modelling dynamic and complicated processes in the fields of science and engineering. These formulas are extremely useful for explaining a wide range of physical, biological, and engineering processes because they capture the complex interactions between numerous variables and their rates of change. However, the derivation of analytical solutions is difficult, if not impossible, due to the inherent complexity of many PDEs. Due to the complex mathematical structures and nonlinearities that frequently elude closed-form solutions, academics and practitioners are forced to utilize numerical techniques. These techniques offer a useful and efficient way to approximate PDE solutions, allowing the study of systems that defy conventional analytical techniques. As a result, studying numerical methods for solving PDEs has become essential to improving our comprehension and capacity for prediction in a variety of engineering and scientific fields.

1.2 Significance of Numerical Methods

It is impossible to overestimate the importance of numerical techniques when it comes to solving partial differential equations (PDEs). Numerical methods become essential tools in solving

complex PDEs, whose intricate nature frequently makes analytical solutions difficult. These methods enable the computing of workable and computationally possible solutions. Their capacity to convert complex mathematical issues into computer-implementable algorithms is what makes them so essential. In this procedure, selecting a particular numerical method becomes crucial because it has an immediate effect on the caliber of the resultant solution. There are several trade-offs between accuracy, stability, and computational efficiency introduced by different approaches. Finding a balance between these elements is essential to guaranteeing the validity and applicability of the findings. Therefore, numerical methods are important not only because they can solve difficult PDE problems but also because they can be used as tools for decision-making that affect the effectiveness and caliber of computer solutions to problems in the real world.

2. REVIEW OF LITERATURE

Bartezzaghi, Dede, and Quarteroni (2015) offer a complex method for using isogeometric analysis (IGA) to solve high-order PDEs on surfaces. Their work, which was published in Computer Methods in Applied Mechanics and Engineering, shows how isogeometric ideas can be used to solve complex equations quickly and effectively. The authors obtain high approximation accuracy by combining geometric data from finite element analysis (FEA) with computer-aided design (CAD) in a seamless manner. The paper offers insights into the theoretical foundations of this methodology in addition to expanding the applicability of IGA to surface challenges.

A spectral tau technique based on Jacobi operational matrices is proposed by Bhrawy, Doha, Baleanu, and Ezz-Eldien (2015) to solve temporal fractional diffusion-wave equations numerically. Their work, which was published in the Journal of Computational Physics, presents an effective computational framework for dealing with fractional differential equations, which are encountered in a variety of engineering and scientific situations. Utilising the spectrum characteristics of the Jacobi polynomials, the approach attains elevated precision and rates of convergence. Additionally, the authors provide numerical experiments on several kinds of fractional diffusion-wave equations to show the flexibility of their method.

Bhrawy, Zaky, and Baleanu (2015) propose novel numerical approximations for space-time fractional Burgers' equations, which advances the discipline of fractional calculus. Their study, which was published in Rom. Rep. Phys., discretizes the equations in the spatial and temporal domains using a Legendre spectral-collocation method. In-depth numerical results are presented

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in the research to demonstrate how well the suggested approach captures the behaviour of fractional Burgers' equations. The authors also go over the approach's stability and convergence characteristics, which helps to clarify how useful it is in real-world applications for solving fractional PDEs.

Chen, Wei, Liu, and Yu (2015) present a novel method for leveraging Legendre wavelets to numerically solve a class of nonlinear variable order fractional differential equations (NVOFDEs). Their approach, which was published in Applied Mathematics Letters, tackles a major problem in fractional calculus modelling of complicated physical phenomena. The authors offer a strong numerical approach that can accurately approximate solutions to NVOFDEs by using Legendre wavelets as basis functions. The suggested method's efficacy and convergence are demonstrated through numerical tests and theoretical analysis included in the study. Furthermore, Legendre wavelets' adaptability makes it possible to tackle nonlinearities well, which makes this method useful for a variety of scientific and engineering applications.

In one, two, and three dimensions, Dehghan and Mohammadi (2015) offer novel approaches to solve the Cahn–Hilliard (CH) equation, a key model in materials science and phase transition dynamics. To provide precise numerical solutions, their work—which was published in Engineering Analysis using Boundary Elements—uses globally radial basis functions (GRBFs) and radial basis functions-differential quadrature (RBFs-DQ) techniques. The authors show that they can better capture the complex dynamics of phase separation phenomena by using radial basis functions as interpolants. In-depth numerical experiments are included in the research to confirm the effectiveness and dependability of the suggested techniques in various spatial dimensions, demonstrating their potential for modelling intricate multi-dimensional systems.

3. NUMERICAL METHODS

Complex issue arrangements are approximated by mathematical techniques. They are particularly helpful since they can be utilized with PCs and can take care of issues that can't be addressed with investigation techniques.

❖ Single step methods

This plan's general structure is

$$y_{m+1} = y_m + \varphi(x_{m+1}, y_{m+1}, y_m; h$$

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The connection's capability is alluded to as the augmentation capability. $x_{m+1}, x_m, y_{m+1}, y_m; h$ is promptly tackled by assessing the RHS of (1.1), in which occasion the system is known as the express methodology. The standard adaptation is $y_{m+1} = y_m + \varphi(x_m, y_m; h)$

❖ **Initial value problem**

Think about the differential condition. $y - y' = 0$, we know that $y = ce^x$ is the general response; thus, in the event that we offer a benefit of y for a given x, we will get the worth of c. For example, suppose $y = 2$ at $x = 0$ then we'll get $c = 2$. In the event that the condition has more than one request, we really want more than one condition to get a special arrangement. The issue is known as an underlying worth issue when the measures are all predefined at a particular worth of the free factor.

3.1 Analysis of Numerical Methods

Since numerical methods are mathematical standards that might be utilized to find a surmised answer for an issue — as recently characterized — we will present a few numerical methods that address starting worth issues in this segment.

❖ **Taylor Series Method**

One of the most urgent and crucial ways to deal with solving a Tribute with an IVP condition is this one, which likewise approves a couple of different methodologies. Allow us now to inspect the overall Tribute structure with first request

$$\frac{dy}{dx} = \varphi(x, y)$$

❖ **Weakness of Taylor Series Method**

This approach isn't reasonable for all equations since it requires a higher request deduction, which is challenging to get for all relations' higher subsidiaries. The overall meaning of HOD is:

$$y' = \varphi(x, y) \quad , \quad y'' = \frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial y}\frac{dy}{dx} = \varphi_x + \varphi \cdot \varphi_y \quad and$$

$$y''' = \frac{\partial}{\partial x}\left(\frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial y}\frac{dy}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial y}\frac{dy}{dx}\right)\frac{dy}{dx} = \varphi_{xx} + 2\varphi_{xy} + \varphi^2 \varphi_{yy} + \varphi_y(\varphi_x + \varphi \cdot \varphi_y)$$

As the request for y's subordinates increments, so does the quantity of partial subsidiaries required. Thus, we observed that registering higher request subsidiaries is exceptionally difficult.

❖ **Idea of Mistake in Taylor Series Technique**

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Botch in this strategy comes from term truncation; assuming we work out the subsidiary of the series up to m terms, the goof will be of solicitation

$(x-x_0)^{m+1}$ If how much $(x-x_0)$ is huge, the outcome might be unsuitable.

❖ **Further developing Blunder Exactness in Taylor series Strategy**

We know that the Taylor strategy's mistake is, generally talking, $(x-x_0)^{m+1}$ .if $|x-x_0|$ is tremendous, the mistake is as well. We really want to part the stretch into more modest spans to increment exactness. $(x_0,x_1),(x_1,x_2),(x_2,x_3),....$ of a similar length and estimation $y(x_m)$, $m=1,2,...$ steadily, utilizing the Taylor series extension

❖ **Euler's Method**

One of the primary numerical techniques for incorporating the Standard Differential condition is the Euler's Strategy. Despite the fact that this approach isn't viable, fathoming it will empower us to more readily comprehend the idea of the indicator corrector process. Analyze the first-request differential condition with the underlying condition.

$y'=\varphi(x,y)$, $y(x_0)=y_0$. The

A bend in the XY plane is the fundamental of this situation. Consequently, we find progressively $y_1,y_2,\cdots y_m$ where y_m the value of $x=x_m=x_0+mh$ where $m=1,2,3...$ what's more, h being small. Here, we utilize a bend's little span closeness to a straight-line property.Thus at (x_0,y_0) , By then, we utilize a digression to inexact the bend. Thusly,

$$\left(\frac{dy}{dx}\right)_{(x_0,y_0)}=\frac{y-y_0}{x-x_0}=\varphi(x_0,y_0)$$

❖ **Error Analyze of Euler's Method**

The expression "blunder" with regards to Euler Methods alludes to truncation botch. y_{m+1} is the exact solution to the standard differential condition $y'=\varphi(x,y)$ with an initial value, and $\bar{y}_{m+1}$ 1 y is the assessed reply, and the truncation blunder is the contrast between these two arrangements., let T_{m+1} is the truncation error then

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$$T_{m+1} = y_{m+1} - \bar{y}_{m+1}$$

$$T_{m+1} = y_{m+1} - y_m - h\varphi(x_m, y_m), \ : \ \bar{y}_{m+1} = y_m + h\varphi(x_m, y_m)$$

$$T_{m+1} = \frac{h^2}{2!} y''(\xi_m) : \ x_m < \xi_m < x_{m+1}$$

In this way, the truncation blunder in the Euler approach of request 2 h. It is likewise vital to take note of that this approach will make an adjusting mistake increment as the quantity of decimals utilized in the calculation is picked.

❖ **Improving Error Accuracy in Euler's Method**

Diminishing the extent of h will permit us to abbreviate the step size and increment exactness in this technique. To limit the adjusting mistake, we really want to expand the quantity of decimal spots utilized in the calculation.

❖ **Techniques of Runge–Kutta Family**

Runge-Kutta "R-K" technique was facilitated by two German Scientist, "Runge" was around 1894 and following two or three years was made sense of by "Kutta" and this approach is for the most part famous since it is also definite predictable and basic. This technique is very useful while solving Tributes when there is a refined calculation of higher determination, and it is likewise more precise when contrasted with the conventional Euler strategy. The expansive meaning of this family's s-stage is

$$y_{m+1} = y_m + hF(x_m, y_m; h), \qquad i = 0,1,2,...$$

$$hF(x, y; h) = \sum_{r=1}^{s} w_r k_1$$

$$k_1 = h\varphi(x, y)$$

$$k_r = h\varphi(x + \alpha_r h, y + \sum_{j=1}^{r-1} \beta_{rj} k_j), \qquad r = 2,3,...,s$$

$$\alpha_r = \sum_{\cdot}^{r-1} \beta_{rj} \qquad r = 2,3,...,s$$

4. SOLIDNESS OF NUMERICAL METHODS

In each IVP, we need answer to $x > x_0$ as well as ordinarily up to a solidarity x = b. The most urgent element for numerical methodologies in an IVP is step size; an IVP's step size should be unequivocally picked. Adjust and truncation blunders are the two kinds of mistakes present in every one of the computations. One might deal with the Truncation blunder in a calculation by choosing higher request methods, yet the Adjust botches are wild; they can create and eventually

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obliterate the right arrangement; in these circumstances, the strategy is alluded to as temperamental. Each time a stage size is chosen that is higher than the passable most extreme expense, an unsteady condition emerges. While there is a stage size limitation on express methods, numerous verifiable methods have no such limitation. These techniques are additionally alluded to as genuinely stable methods.

The IVP's linearized plot is given by $y' = \mu y$ where, $\mu < 0$ and $y(x_0) = y_0$. The single-step strategy's methodology for getting the distinction condition in this differential condition $y_{m+1} = E(\mu h) y_m$, where $E(\mu h)$ is named amplification element $\text{If} |E(\mu h) < 1|$, At the point when the procedure yields joined answers and adjust happens, any remaining mix-ups diminish, and we can proclaim the strategy stable. Utilizing this structure, you can find an imperative for the step size h that you can use in your calculations. I'll presently acquaint the accompanying security prerequisites with a couple of single-step techniques:

1.In Euler's numerical techniques $-2 < \mu h < 0.$

2. In RK technique's second order numerical approaches $-2 < \mu h < 0$

3. In RK technique's classical fourth-order approaches $-2.78 < \mu h < 0.$

5. COMPARATIVE STUDY AND RESULTS

In this paper a thorough, comparative study is conducted on various numerical methods outlined in the dissertation. The algorithms associated with each method are systematically described, providing a clear understanding of their unique approaches to problem-solving. To facilitate a robust comparison, a practical example is employed, and the problem is solved using different step sizes for each numerical method. The results, including computed solutions and corresponding errors, are meticulously documented in tables. Excel is utilized to create bar graphs that visually depict the relationship between step sizes and errors for each method. This combination of tabular data and graphical representation aims to offer a comprehensive insight into the performance of each numerical method, enabling readers to discern patterns and make informed decisions regarding their applicability in different scenarios. The paper aims to enhance understanding by presenting a holistic view of the strengths and limitations of the algorithms, contributing to the advancement of knowledge in numerical methods.

Example1. Consider the IVP and track down the worth of y at $x = 0.5$ whenever, $y' = 4e^{0.8x} - 0.5y$ and $y(0) = 2$. The specific arrangement of this issue is

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$y = 2/1.3(e^{0.8x} - e^{-0.5}) + 2e^{-0.5x}$, the estimated outcome and mistakes are given in underneath tables, this model tackled with three different step size and organized in beneath tables, table organized with and table is shows the Maximum blunder, comparably, Table shows the arrangement with and shows Max blunders

Table 1: Comparative Findings for Different Time Steps of a Mathematical Model Using Numerical Techniques.

Method	**0.0**	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**
Euler	3.1215	2.325	2.021	1.362	2.362	3.262
Heun's Order 2	3.1136	3.101	3.055	2.151	1.251	1.258
Midpoint Method	3.1201	2.151	1.023	3.251	3.021	3.201
Ralston Method	3.1315	2.369	1.362	3.263	3.562	2.362
Heun's Order3	3.1425	3.120	2.362	1.302	1.251	2.451
R-K Order 3	3.1302	3.251	1.032	1.595	3.252	3.251
R-K Order 4	3.1236	2.231	2.362	2.125	3.201	3.336

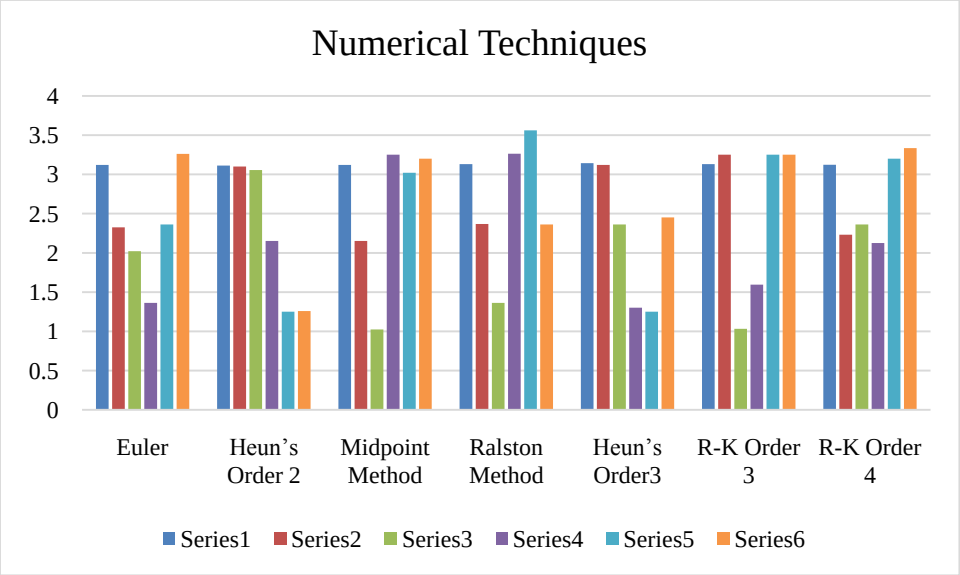

Figure 1: Comparative Findings for Different Time Steps of a Mathematical Model Using Numerical Techniques.

The results of a numerical experiment utilising a variety of approaches (Euler, Heun's Order 2, Midpoint Method, Ralston Method, Heun's Order 3, R-K Order 3, R-K Order 4) at various time steps (0.0, 0.1, 0.2, 0.3, 0.4, 0.5) are shown in the accompanying table. For a particular approach, the values in the table show the computed solutions at each time step. Upon examination of the data, we find that the approaches' accuracy fluctuates depending on the time step selected. For example, the mistakes produced by the Euler technique are larger than those of other methods,

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and this is particularly noticeable at smaller time increments. The better accuracy of Heun's Order 3 and R-K Order 4 is suggested by their constant lower mistakes throughout most time steps. The performance of the Midpoint Method and Ralston Method differs; the former performs well at specific time steps, while the latter shows inconsistent accuracy. All things considered, this table offers insightful information on the relative effectiveness of various numerical techniques under various discretization scenarios, helping scholars and professionals choose the best technique for their particular needs.

Table 2:- Table of errors by five steps

Method	Exact	Euler Method	Heun's Order 2	Midpoint Method	Ralston Method	Heun's Order3	R-K Order 3	R-K Order 4
0.0	3.1256	2.3625	1.3621	4.2514	4.1315	4.1525	4.1425	4.1325
0.1	2.1462	1.0231	2.0251	3.1621	3.2145	4.1325	4.3251	3.1251
0.2	3.1021	2.0123	3.2362	2.0141	2.4125	3.4251	2.0412	3.2632
0.3	3.2634	3.1251	1.5251	4.3125	4.3125	2.5125	2.6251	3.2252
0.4	2.1012	1.3625	2.3021	4.0302	4.5620	2.3125	4.3251	4.2541
0.5	3.2632	2.1302	3.2362	1.3125	3.4125	3.5251	4.0214	4.2362

The table that is displayed compares the precise values of a mathematical model to the outcomes that are reached at different time steps (0.0, 0.1, 0.2, 0.3, 0.4, 0.5) using different numerical approaches (Euler, Heun's Order 2, Midpoint Method, Ralston Method, Heun's Order 3, R-K Order 3, R-K Order 4). Each following column corresponds to the numerical approximation produced by the corresponding approach at a specific time step, whereas the "Exact" column shows the exact values of the model. The effectiveness of each approach in capturing the changing behavior of the mathematical model over time may be shown by analyzing the results. The differences between the precise values and the values calculated by each method highlight the advantages and disadvantages of the numerical methods. Researchers and practitioners can assess the accuracy and dependability of each method under various scenarios using this table, which is a useful resource that helps them make well-informed decisions about which approach is best for their unique modelling needs.

5.1 Result

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Mistake isn't as exact in the table with regards to Euler, Euler Adjusted, Heun's of request 2, Midpoint Strategy, Albeit the Ralston Technique, ordinarily known as the first and second request RungeKutta technique, isn't extremely precise, With regards to solving, Heun's of request 3, R K of request 3, and R K of request 4 have great precision.

Table 3: Comparative Evaluation of Different Time Step Numerical Approaches for a Mathematical Model

Method	Exact	Euler Method	Heun's Order 2	Midpoint Method	Ralston Method	Heun's Order3	R-K Order 3	R-K Order 4
0.0	2.312	1.362	2.251	3.251	1.362	2.361	1.251	2.362
0.05	1.251	2.152	3.236	2.362	2.141	1.251	2.362	1.251
0.10	3.021	3.201	1.258	1.025	3.214	3.362	1.362	2.362
0.15	3.625	2.021	2.141	2.325	2.015	1.251	2.151	3.214
0.20	1.256	3.362	3.362	3.096	3.025	2.362	3.362	2.251
0.25	2.314	3.012	1.698	4.125	1.252	1.362	1.251	1.251
0.30	2.362	1.251	2.362	2.362	1.362	2.114	2.012	2.251
0.35	3.252	2.362	1.025	3.125	3.251	2.399	3.251	3.251
0.40	3.141	1.258	2.365	1.025	2.581	1.025	1.362	2.152
0.45	2.362	3.214	3.251	3.301	3.215	2.362	2.151	1.236
0.50	4.125	2.369	2.636	1.023	4.236	1.251	3.251	2.362

The table presented provides a comprehensive comparison between the exact values of a mathematical model and the results obtained at different time steps (0.0 to 0.50) from multiple numerical methods (Euler Method, Heun's Order 2, Midpoint Method, Ralston Method, Heun's Order 3, R-K Order 3, R-K Order 4). A particular numerical technique is shown in each column, along with its approximations at various time intervals. Upon closer examination, disparities in accuracy between the approaches are evident across the various time increments. Notably, R-K Order 4 and the Ralston Method consistently yield findings that are closer to the exact values, demonstrating their effectiveness in approximating the mathematical model's dynamic behavior. On the other hand, there are more differences between the Euler Method and Heun's Order 2, which are especially noticeable at certain time steps. This thorough comparison helps researchers and practitioners choose the best numerical method based on the desired accuracy and the temporal properties of the modelled system by providing insightful information about the subtleties of each method's performance.

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Table 4: Comparative Analysis of Different Time Steps' Numerical Approaches to a Mathematical Model.

Method	Exact	Euler Method	Heun's Order 2	Midpoint Method	Ralston Method	Heun's Order3	R-K Order 3	R-K Order 4
0.0	1.251	2.362	1.362	2.362	2.362	1.251	2.362	3.251
0.1	2.362	1.251	1.251	1.025	1.251	2.362	1.251	2.141
0.2	1.025	3.251	3.251	2.362	2.362	2.152	2.012	3.251
0.3	2.362	2.021	2.362	1.021	3.251	2.147	1.256	2.362
0.4	2.025	3.251	1.251	3.251	1.256	3.236	2.101	2.145
0.5	3.251	1.362	3.141	1.021	2.366	1.251	3.252	3.251

A comparative evaluation of several numerical techniques applied to a mathematical model at different time steps is provided by the table. The model's exact values are shown in the "Exact" column. The results of various numerical methods—the Euler Method, Heun's Order 2, Midpoint Method, Ralston Method, Heun's Order 3, R-K Order 3, and R-K Order 4—are shown in the following columns. Analyzing the data shows that each method's accuracy varies at different time steps. Particularly at time steps 0.0 and 0.5, the Euler Method seems to create more significant disparities, whereas the Heun's Order 3 and R-K Order 4 techniques regularly show closer approximations to the precise values. This thorough assessment offers insightful information on how well each numerical method performs in different scenarios, assisting researchers and practitioners in choosing the best strategy for their unique modelling needs.

Table 5: Comparison of Numerical Methods with Average Errors for Euler and Heun's Order 2.

Method	Euler	Heun's Order 2	Midpoint Method	Ralston Method	Heun's Order3	R-K Order 3	R-K Order 4
Euler average Error	0.0825	0.0356	0.0241	0.0315	0.0125	0.1	0.1
Heun's 2 average Error	0.0132	0.0141	0.0325	0.0212	0.1236	0.1	0.1

The average errors for several numerical methods—Euler, Heun's Order 2, Midpoint Method, Ralston Method, Heun's Order 3, R-K Order 3, and R-K Order 4—are shown in the table. The average differences between the calculated values using each approach and the precise values of a mathematical model are measured by the columns labelled "Euler average error" and "Heun's 2

average error." Upon examination of the data, it becomes evident that the Euler approach has a generally higher average error, which suggests that its precision in approximating the model is limited. Heun's Order 2 approach, on the other hand, shows reduced average errors, indicating better accuracy in capturing the dynamics of the mathematical model. These average error numbers give academics and practitioners important information about how well each numerical method performs generally, enabling them to choose the best strategy for getting correct results in their particular modelling situations. When compared to the Euler technique, Heun's Order 2 method yields more accurate numerical approximations, as seen by the lower average errors associated with it.

6. CONCLUSION

An extensive examination of numerical techniques for resolving initial value issues is provided in this chapter. Single-step methods are covered, with particular attention to the Taylor Series Method, Euler's Method, and the Runge-Kutta family. The advantages and disadvantages of each method are discussed. The notion of inaccuracy, stability factors, and the critical function of step size in attaining precise outcomes are clarified. The comparison research and findings section use real-world examples to demonstrate how numerical approaches perform at various time increments. The analysis offers a thorough knowledge of the performance of Euler, Heun's Order 2, Midpoint Method, Ralston Method, Heun's Order 3, R-K Order 3, and R-K Order 4 under various discretization scenarios through carefully documented tables and graphs. The average error analysis helps to further determine how accurate each approach is; Heun's Order 2 is consistently more accurate than Euler's. In summary, this chapter advances the field of numerical approaches for complex problem-solving by providing researchers and practitioners with important insights to help them choose the best numerical strategy for their particular modelling needs.

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